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PLANE PERTURBED MOTION OF A MATERIAL POINT OF VARIABLE MASS

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Motion of a material point of variable mass in a central force field in the presence of a perturbing force is considered. The solution of the problem is obtained in quadratures.

1. We consider a motion of a material point of variable mass in a central perturbing field obeying the following law

$$P(r) = \lambda m r^{-n} \tag{1.1}$$

where λ and n are the constant field characteristics (when $n = 2$ and $\lambda < 0$, we have the case of a Newtonian gravitational field), m is the mass of the material point and r is its distance from the center.

We assume that the mass of the point is a continuously differentiable function of its distance from the center

$$m = m_0 f(r), \qquad r = r(t) \qquad (m = m_0,\ r = r_0, \text{ for } t = 0) \tag{1.2}$$

and we also assume that

$$\mathbf{u} = p(r)\,\mathbf{v} \tag{1.3}$$

where $\mathbf{v}$ is the velocity of motion of the point in the inertial frame of reference, $\mathbf{u}$ is the velocity of the particles rejected (or assimilated) by the parent point up to the given instant and $p(r)$ is a specified continuous function. Then the reaction force can be written as

$$\mathbf{R}(r) = m_0 r^{\cdot} f'(r)\, g(r)\, \mathbf{v}, \qquad g(r) = p(r) - 1 \tag{1.4}$$

where a dot denotes derivative with respect to time and a prime, with respect to r. We assume that in addition to the forces given, a perturbing force $\mathbf{F}^*$ lying in the plane of the trajectory and orthogonal to the vector $\mathbf{v}$ is also acting on the point. An analogous problem of motion of a point of constant mass was studied in [1], where it was shown that if

$$\mathbf{F}^* = m_0 \mathbf{F}(r, v) \tag{1.5}$$

then the problem can be reduced to quadratures. We shall show that this remains true for the case of a point of variable mass.

Equations of plane motion of a point under the conditions (1.1) - (1.3) have the following form in the polar coordinates:

$$r^{\cdot\cdot} - r\varphi^{\cdot 2} = \frac{\lambda}{r^n} + \frac{1}{f} F_r + \frac{f'}{f} g r^{\cdot 2}, \qquad (r^2\varphi^{\cdot})^{\cdot} = \frac{r}{f} F_\varphi + \frac{f'}{f} g r^2 r^{\cdot} \varphi^{\cdot} \tag{1.6}$$

$$F_r = -F(r, v)\frac{r\varphi^{\cdot}}{v}, \qquad F_\varphi = F(r, v)\frac{r^{\cdot}}{v}, \qquad v^2 = r^{\cdot 2} + r^2\varphi^{\cdot 2}$$

Let us find the solution of (1.6) for initial conditions

$$r = r_0, \quad r^{\cdot} = r_0^{\cdot}, \quad \varphi = \varphi_0, \quad \varphi^{\cdot} = \varphi_0^{\cdot} \quad \text{for} \quad t = 0 \tag{1.7}$$

Eliminating F we obtain

$$\frac{dv^2}{dr} - 2g\frac{f'}{f}v^2 = \frac{2\lambda}{r^n} \tag{1.8}$$

and this yields the energy integral

$$v(r) = \mu\left(v_0^2 + 2\lambda\int_{r_0}^{r}\frac{dr}{\mu^2 r^n}\right)^{1/2} \equiv G(r)$$

$$\mu = \exp\left(\int_{r_0}^{r}\frac{f'}{f}g\,dr\right), \qquad v_0^2 = r_0^{\cdot 2} + r_0^2\varphi_0^{\cdot 2} \tag{1.9}$$

The second equation of (1.6) can be written as

$$(r^2\varphi^{\cdot})' - g\frac{f'}{f}(r^2\varphi^{\cdot}) = \frac{r}{f}\frac{H(r)}{G(r)}$$

$$H(r) = F(r,\ G(r))$$

and this gives the expression analogous to an area integral

$$r^2\varphi^{\cdot} = \mu\left[r_0^2\varphi_0^{\cdot} + \int_{r_0}\frac{H(r)}{G(r)}\frac{r\,dr}{\mu f}\right] \equiv \Phi(r) \tag{1.10}$$

Eliminating φ and the relation for v, we obtain

$$t = \sigma\int_{r}^{r}(G^2 - \Phi^2 r^{-2})^{-1/2}dr \qquad (\sigma = \pm 1) \tag{1.11}$$

The integral (1.11) exists for any continuous smooth trajectories except the circular ones. The signs σ and λ are chosen with regard to the initial conditions and to the character of the field.

From the integrals (1.10) and (1.11) we obtain

$$\varphi(r) = \varphi_0 + \sigma\int_{r_0}^{r}\Phi r^{-1}(G^2r^2 - \Phi^2)^{-1/2}dr \tag{1.12}$$

The values of r_1 and r_2 for which $v_r = 0$ are poles of the integrand functions in (1.12) and (1.11). The motion of the point takes place, in general, within the annulus $r_1 < r < r_2$ and the case in which $r_2 \to \infty$ [1] is possible. The equations (1.11) and (1.12) determine the law of the plane motion of a point of variable mass.

2. We assume that the function of mass flow

$$m^{\cdot} = m_0\eta(t), \qquad \eta[t(r)] = q(r) \tag{2.1}$$

continuous for all t is specified. Then, using the relations (1.2) and (1.11) we obtain

$$f(r) = 1 + \sigma\int_{r_0}^{r}qr(G^2r^2 - \Phi^2)^{-1/2}dr \tag{2.2}$$

Here the sign of σ determines two modes of the change in mass, the flow and the addition.

Using the results obtained we can solve the problem of determining the law governing the mass change employing the prescribed manner of variation of the modulus of its velocity given as a function of its distance from the center

$$v^2 = s(r) \tag{2.3}$$

From (1.8) we have

$$f(r) = \exp\left\{\frac{1}{2}\int_{r_0}^{r}\left[s'(r) - \frac{2\lambda}{r^n}\right]\frac{dr}{gs}\right\} \tag{2.4}$$

If the program of variation in the sectorial velocity and the radial velocity component of the point

$$r^2\varphi^{\cdot} = k(r), \qquad v_r = u(r) \tag{2.5}$$

are both specified then (1.8) yields the law of mass change in the form

$$f(r) = \exp\int_{r_0}^{r}\psi(r)\,dr \tag{2.6}$$

$$\psi(r) = \frac{1}{g(r)}\frac{r^2}{u^2r^2+k^2}\left[-\frac{\lambda}{r^n} + uu' + \frac{k}{r^2}\left(k' - \frac{k}{r}\right)\right]$$

If on the other hand the relation (2.3) and the first relation of (2.5) both hold, then the function of mass change is

$$f(r) = \frac{rH^*(r)}{\sqrt{s(r)}}\left[k' + \left(\frac{\lambda}{sr^n} - \frac{s'}{2s}\right)k\right]^{-1}$$

$$H^*(r) = F\left[r, \sqrt{s(r)}\right] \tag{2.7}$$

When $p(r) =$ const, Eq. (2.6) exists only for the values of $p \neq 1$.

The relationships determining the solution of the converse problem are of interest: given the characteristic functions $G(r)$ and $\Phi(r)$ of the motion of the point, to find the perturbind forces. From (1.10) we find

$$r^{\cdot} = \pm r^{-1}\Delta(r), \qquad \Delta(r) = (r^2G^2 - \Phi^2)^{1/2}$$

Defining the quantity r, we obtain from the first equation of (1.6)

$$F_r = \left[r^2\frac{d}{dr}\left(\frac{G^2}{2}\right) - \frac{d}{dr}\left(\frac{\Phi^2}{2}\right) - \frac{f'}{f}g\Delta^2 - \frac{\lambda}{r^{n-2}}\right]\frac{f(r)}{r^2} \tag{2.8}$$

Similarly, the second equation of (1.6) yields

$$F_\varphi = \pm\left(\Phi' - \frac{f'}{f}g\Phi\right)\Delta(r)\frac{f(r)}{r^2} \tag{2.9}$$

Formulas (2.8) and (2.9) represent the generalized analogs of the Binet formulas in the theory of central motion of a point. Here the function $f(r)$ is specified.

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